The mode-coupling Liouville–Green approximation for a two-dimensional cochlear model

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The Liouville–Green [or Wentzel–Kramers–Brillouin (WKB)] approximation for the two-dimensional cochlear mechanics problem disagrees with the finite-difference solution in the region after the response peak. This disagreement has left doubts about the validity of the Liouville–Green approximation, and has never been satisfactorily explained. In this paper, it is shown that the Liouville–Green approximation fails to satisfy Laplace's equation. A new solution is proposed, called the *mode-coupling Liouville–Green approximation*, in which energy is coupled into a second wave mode, so as to obey Laplace's equation. The new approximation gives excellent quantitative agreement with the finite-difference solution. Furthermore, it may provide an explanation for a second vibration mode observed in biological cochleas. Also proposed is a high-order formulation of the stapes displacement term, which is necessary to obtain good agreement between the Liouville–Green approximation and finite-difference solutions at low frequencies. [S0001-4966(00)05010-4]

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I. INTRODUCTION

The Liouville–Green (LG), or WKB, approximation to the cochlear mechanics problem has been studied extensively prior to the mid-1980's by Zweig *et al.* (1976), Steele and Taber (1979), and de Boer and Viergever (1982). However, the advent of fast computers and efficient numerical algorithms has prompted many researchers to abandon the analytic, approximate LG method in favor of "exact" numerical methods such as the finite-difference (FD) and finite-element (FE) methods. These exact solutions allow efficient computation of the response to a given stimulus, and sometimes expose unexpected behavior. Unfortunately, they do not provide insights into the complex wave dynamics that can be provided by an accurate analytic expression for the solution.

In 1979, Steele and Taber compared the two-dimensional Liouville–Green approximation to the finite-difference solution of Neely (1981), as shown in Fig. 2. The LG approximation is an excellent approximation to the FD solution up to a few millimeters past the response peak, at which point the two solutions suddenly start to diverge. This discrepancy was addressed directly by de Boer and Viergever (1982); they stressed that the eikonal equation in the LG analysis has multiple roots, and noted that the change in slope in the amplitude response curves corresponds to the sudden emergence of a second wave mode. However, they were unable to predict accurately when the second wave mode would emerge. This uncertainty about the validity of the Liouville–Green approximation also contributed to its falling out of favor in the hearing research community.

In this paper, it will be shown that the Liouville–Green approximation fails to satisfy Laplace's equation. A new approximation, called the mode-coupling Liouville–Green approximation, is proposed which is a linear superposition of the usual traveling wave mode and a second cutoff mode. The amplitude of the second wave mode is determined by requiring that Laplace's equation be satisfied on average in a given vertical slice of the fluid. This procedure leads to an accurate prediction of the emergence of the second wave mode, and produces an approximation that agrees quantitatively with the finite-difference solution. Moreover, the new approximation provides an explanation for a second wave mode first observed in squirrel monkey basilar membrane responses (Rhode, 1971).

In this paper, we also introduce a high-order formulation for the stapes displacement, which is necessary to obtain good agreement between the Liouville–Green approximation and numerical solutions at low frequencies.

II. THE LIOUVILLE–GREEN APPROXIMATION

The two-dimensional cochlear model is shown in Fig. 1, where $\phi(x,y,t)$ is the velocity potential, ρ is the fluid density, h is the height of the duct, L is the length of the duct, and $S(x)$, $\beta(x)$, and $M(x)$ are the stiffness, damping, and mass, respectively, of the basilar membrane (BM). The BM is located at $y=h$. The fluid is assumed to be incompressible.

The Liouville–Green approximation of the 2D cochlear model has been derived previously in the literature (Steele and Taber, 1979), and is the starting point for the present discussion. For brevity, only the major results are shown below.

The approximation for the velocity potential for a sinusoidal input with frequency ω is given by

$$\phi(x,y,t)=\frac{C\omega\cosh(ky)}{\cosh(kh)\sqrt{\tanh kh+kh\,\mathrm{sech}^2\,kh}}\times\exp\left(i\omega t-i\int_0^x k(u)\,du\right), \quad (1)$$

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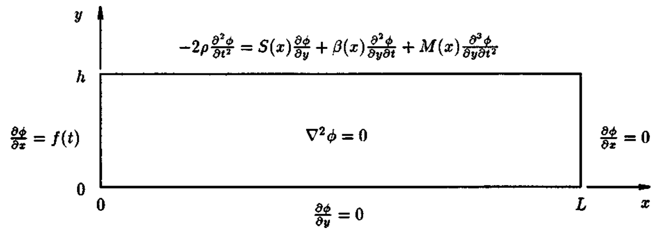

FIG. 1. The two-dimensional cochlear model.

where C is a constant proportional to the stapes displacement, and $k(x)$ is the wave number, which is found by solving the dispersion relation (or eikonal equation)

$$k(x)\tanh k(x)h = \frac{2\rho\omega^2}{S(x)+i\beta(x)\omega - M\omega^2}. \tag{2}$$

The vertical displacement, δ, of the basilar membrane is related to the vertical velocity, $\mathbf{v}_y$, of the fluid at $y=h$, and the velocity potential, by the following relations:

$$\frac{\partial\delta}{\partial t} = \mathbf{v}_y = -\frac{\partial\phi}{\partial y}. \tag{3}$$

A high-order formulation for the stapes displacement d_{st} has been determined by Watts (1992) to be

$$d_{\mathrm{st}} = \frac{C\mathrm{T}(k_0,k_0')\exp(i\omega t)}{h\sqrt{\tanh(k_0h)+k_0h\,\mathrm{sech}^2(k_0h)}}, \tag{4}$$

where

$$T(k_0,k_0') = \tanh(k_0h)\left[1-\frac{ik_0'}{k_0^2} - \frac{2ik_0'h(1-k_0h\tanh(k_0h))}{k_0(2k_0h+\sinh(2k_0h))} - \frac{ik_0'h\tanh(k_0h)}{k_0}\right] + \frac{ik_0'h}{k_0}, \tag{5}$$

where k_0 and k_0' are the values of k and k', respectively, at $x=0$, and the apostrophe indicates differentiation with respect to x. This high-order formulation for the stapes displacement is the key to obtaining a good agreement with the finite-difference solutions at low frequencies.

Finally, the displacement ratio $D(x,\omega)$ can be shown to be

$$D(x,\omega) = \frac{\delta}{d_{\mathrm{st}}} = ikh\frac{\tanh(kh)}{T(k_0,k_0')} \times\sqrt{\frac{\tanh(k_0h)+k_0h\,\mathrm{sech}^2(k_0h)}{\tanh(kh)+kh\,\mathrm{sech}^2(kh)}} \times\exp\left(-i\int_0^x k(u)\,du\right). \tag{6}$$

The magnitude and phase of the displacement ratio in Eq. (6) are plotted in Fig. 2 for the standard Liouville–Green approximation and the finite-difference solution. For comparison with previously published results, we use the parameters of Neely (1981) as shown in Table I.

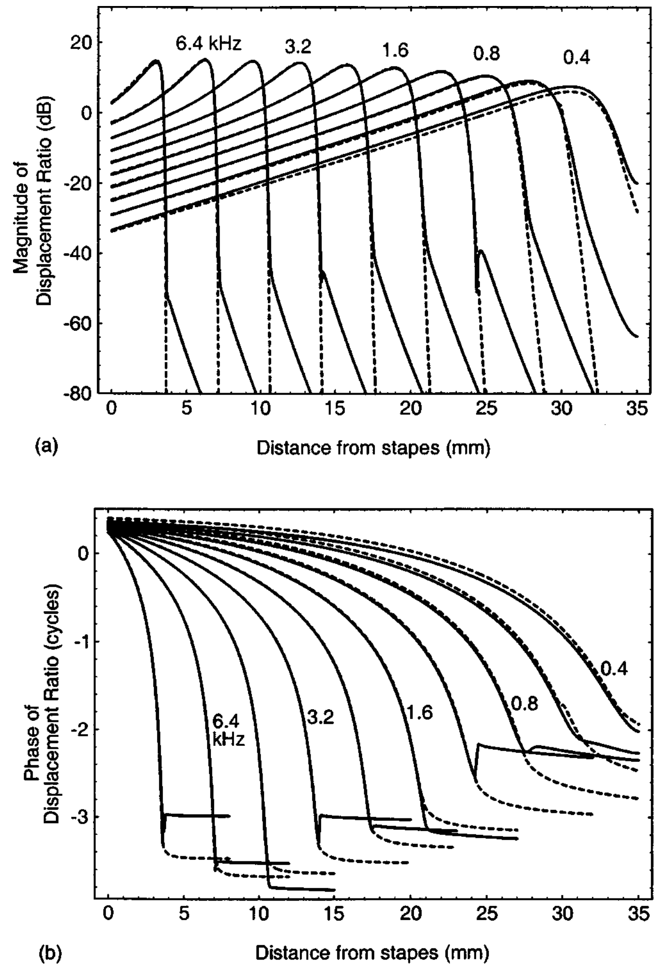

FIG. 2. Comparison of Liouville–Green (dashed) and finite- difference (solid) methods, recomputed after Steele and Taber (1979), and Neely (1981), using the high-order stapes correction of Watts (1992). The two solutions have good agreement up to and just after the peak response. The complete list of frequencies used, in kHz, is (from left to right in the plots): 9.05, 6.4, 4.53, 3.2, 2.26, 1.6, 1.13, 0.8, 0.57, and 0.4.

Note the excellent agreement up to the peak response, and the disagreement just after the peak response.

III. THE LIOUVILLE–GREEN APPROXIMATION FAILS TO SOLVE LAPLACE'S EQUATION

In this section, we shall see that the Liouville–Green approximation associated with the primary root $k(x)$ fails to satisfy Laplace's equation in the fluid just basalward of the cutoff region.

The LG approximation for the velocity potential was designed to satisfy Laplace's equation—that is, to ensure that the flow into any region of space in the x direction is exactly canceled by the flow out of the region in the y direction.

TABLE I. Neely's parameters.

$S(x)=S_0\exp(-x/d)$
$S_0=1.0\times10^7$ g s^{-2} mm^{-2}
$\beta=2.0$ g s^{-1} mm^{-2}
$M=1.5\times10^{-3}$ g mm^{-2}
$d=5.0$ mm
$h=1.0$ mm
$L=35.0$ mm
$\rho=1.0\times10^{-3}$ g mm^{-3}

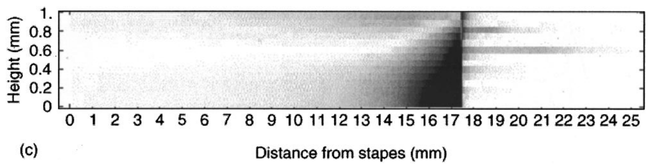

FIG. 3. Relative Laplace error (RLE) shown as a density plot as a function of position in the duct, for Neely's parameters with $f=2.26\,\mathrm{kHz}$. White corresponds to RLE$=0$; black corresponds to RLE>1.

Since the LG approximation is only approximate, we do not expect that $\nabla^2\phi$ is exactly zero; we expect only that the net accumulation or loss in the region is small compared to the amount flowing through it, for a small relative error. Expressed quantitatively, the solution obeys Laplace's equation if

$$|\nabla^2\phi| \ll |k^2\phi| \quad \text{or} \quad \left|\frac{\nabla^2\phi}{k^2\phi}\right| \ll 1. \tag{7}$$

We shall refer to the term $|\nabla^2\phi/k^2\phi|$ as the *relative Laplace error*, or RLE, and we shall refer to inequality (7) as the RLE *criterion*. A nonzero value of RLE implies that the assumption of fluid incompressibility is being violated.

It can be shown that the RLE criterion reduces to

$$\left|\frac{k'}{k^2}\left(1-\frac{4kh(1-kh\tanh kh)}{2kh+\sinh 2kh}+2k(y\tanh ky-h\tanh kh)\right)\right| \ll 1. \tag{8}$$

Note that the RLE criterion has different properties than the conventional validity criterion $|k'/k^2|\ll 1$. The relationship between the RLE criterion and the conventional validity criterion is discussed in the Appendix.

As an illustrative example, let us arbitrarily use Neely's parameters with an input frequency of $f=2.26\,\mathrm{kHz}$. In Fig. 3(a), the relative Laplace error is shown as a density plot as a function of position in the duct. The large black region indicates where Laplace's equation is being violated. The violation becomes significant initially at the bottom of the duct at about $x=15\,\mathrm{mm}$, and rises toward the basilar membrane until about $x=17.4\,\mathrm{mm}$, which is approximately the location at which the corresponding membrane displacement response in Fig. 2 bends. Thus, it can be seen that the unexpected bend in the membrane displacement response is the result of a gradual process that grows from the bottom of the duct over a 2-mm distance in the x direction. This region corresponds to the void left when the wave makes the transition from long-wave to short-wave behavior, thus ''lifting off'' the bottom of the duct.

Recall that the wave number $k(x)$ is the solution of the dispersion relation, which in fact has infinitely many solutions. Typical wave number trajectories are shown in Fig. 4 for three different frequencies. Each curve illustrates one particular wave mode (corresponding to one particular solution of the eikonal equation). We will define the traveling wave mode as the one whose wave number begins near the origin, and we will define the cutoff mode as the one whose wave number ends near $-i\pi/2$. Note that for the examples of $f=800$ and 1131 Hz, the traveling wave mode wave number encircles the cutoff mode wave number, whereas for $f=400\,\mathrm{Hz}$, the traveling wave mode is also the cutoff mode.

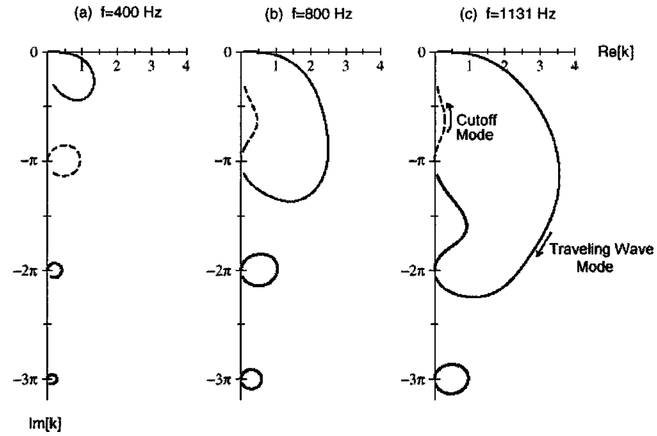

FIG. 4. Example wave number trajectories at three different frequencies, using Neely's parameters. The traveling wave mode begins near the origin. The cutoff mode ends near $-i\pi/2$.

Viergever (1981) observed that the moderate slope in the amplitude and phase responses after the response peak was consistent with the lightly damped cutoff mode. He also observed that the bend in the amplitude curve appeared near the resonance point, so he proposed splicing together the traveling wave and cutoff solutions at the resonance point, prescribing continuity of basilar membrane velocity at the splice. Unfortunately, this procedure resulted in a constant error with respect to the finite-difference solutions.

IV. THE MODE-COUPLING LIOUVILLE–GREEN (MCLG) APPROXIMATION

We now propose the following form of the velocity-potential solution:

$$\phi(x,y,t)=\phi_1(x,y,t)+c(x)\phi_2(x,y,t), \tag{9}$$

where ϕ_1 is the traveling wave solution with wave number k_1, which originates near $k_1\approx 0$ for $x=0$, and ϕ_2 is the traveling wave solution with wave number k_2, which originates near $k_2\approx -(i\pi/2)$ for $x=0$, and $c(x)$ is the coupling coefficient. ϕ_1 and ϕ_2 have been determined already, so all that remains is to determine $c(x)$ such that the composite solution satisfies Laplace's equation.

For Laplace's equation to hold, we must have

$$\nabla^2\phi=\nabla^2\phi_1+c\nabla^2\phi_2+2\frac{\partial c}{\partial x}\frac{\partial\phi_2}{\partial x}+\frac{\partial^2 c}{\partial x^2}\phi_2=0. \tag{10}$$

This equation implies that c must also depend on y to make $\nabla^2\phi(x,y)$ vanish at every point. However, a good approximate solution is possible with $c=c(x)$ alone, so let us specify that the total error must vanish in a vertical slice

$$\int_0^h \nabla^2\phi\,dy=\int_0^h\nabla^2\phi_1\,dy+c(x)\int_0^h\nabla^2\phi_2\,dy +2c'(x)\int_0^h\frac{\partial\phi_2}{\partial x}dy+c''(x)\int_0^h\phi_2\,dy=0. \tag{11}$$

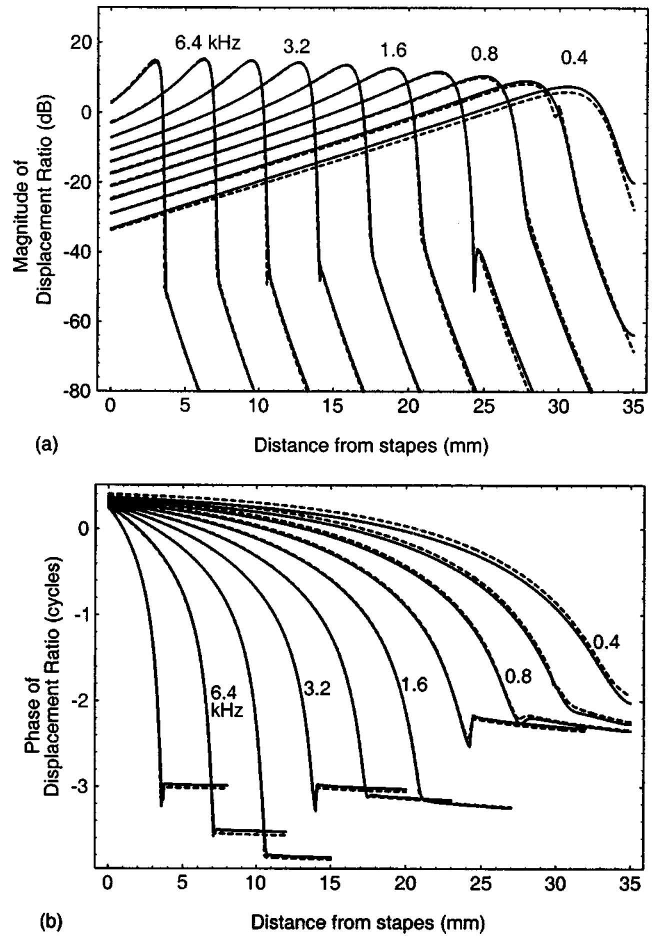

FIG. 5. Comparison of mode-coupling Liouville–Green (dashed) and finite-difference (solid) methods.

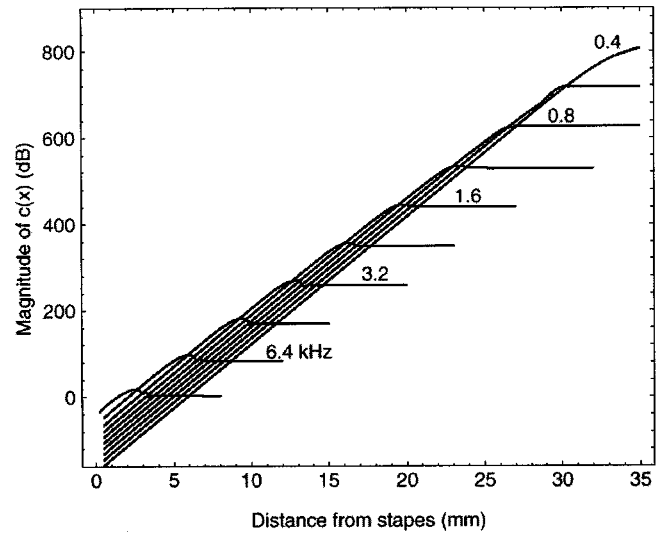

FIG. 6. Magnitude of coupling coefficients $c(x)$ in dB for Neely's parameters. Input frequencies are shown in kHz.

This equation has the form

$$c''(x)+P(x)c'(x)+Q(x)c(x)=R(x), \tag{12}$$

that is, it is a second-order ordinary differential equation in $c(x)$, with nonconstant coefficients given by

$$P(x)=\frac{2\int_0^h(\partial\phi_2/\partial x)\,dy}{\int_0^h\phi_2 dy}, \tag{13}$$

$$Q(x)=\frac{\int_0^h\nabla^2\phi_2 dy}{\int_0^h\phi_2 dy}, \tag{14}$$

$$R(x)=-\frac{\int_0^h\nabla^2\phi_1 dy}{\int_0^h\phi_2 dy}. \tag{15}$$

It is possible to obtain approximate closed-form expressions for the preceding integrals; the lengthy formulas are given by Watts (1992). The boundary conditions for the problem are $c(x)=0$ at $x=0$ and $c'(x)=0$ at $x=L$. Thus, we have a one-dimensional boundary-value problem in $c(x)$ with nonconstant coefficients, which may then be solved numerically for $c(x)$.

V. RESULTS AND DISCUSSION

The mode-coupling Liouville–Green (MCLG) approximation is shown in Fig. 5. The MCLG approximation shows good agreement with the numerical solution in both magnitude and phase of the displacement ratio responses for all frequencies. Clearly, the procedure has predicted the correct amount of energy to couple into the k_2 solution, resulting in good agreement with the numerical solution.

The magnitude of the corresponding coupling coefficients $c(x)$ is shown in Fig. 6. Although the coupling coefficients increase dramatically, they are primarily balancing the natural decay of the cutoff mode. Closer examination of the solutions by Watts (1992) shows that the cutoff mode makes a negligible contribution to the amplitude of the solution at the basilar membrane until the traveling wave mode begins to decay sharply. At this point, the cutoff mode has accumulated significant energy, which it dissipates gradually after the response peak.

In order for the solution to truly satisfy Laplace's equation, it would require contributions from all of the wave modes, not just the traveling-wave and cutoff modes. However, the other modes, by definition, are more heavily damped than the traveling-wave and cutoff modes in their respective regions of dominance, and thus they have only a small local effect which decays quickly after the best place.

The precise relative phase of the two modes in the MCLG approximation is a sensitive function of the physical parameters and the input frequency, and in general may take on any value. Occasionally, the two modes may be exactly out of phase at the basilar membrane position where their amplitudes are equal. In such a case, destructive interference will occur, resulting in a noticeable notch in the amplitude response in the MCLG approximation, as seen, for example, in Fig. 5(a) in the 4.53-kHz trace (third dotted line from the left). In this particular example, the MCLG approximation shows a notch while the numerical solution at that frequency does not, whereas at 1131 Hz [fourth curve from the right in Fig. 5(a)] the numerical solution shows a large notch while the MCLG approximation does not. Clearly, then, both solutions are capable of producing a notch, but they appear to disagree as to when a notch will occur. Since the MCLG solution is only approximate, small errors in the relative phase of the traveling-wave and cutoff modes may cause the notch to appear or disappear in disagreement with the numerical solution. So, the MCLG approximation provides a mechanism and interpretation of the notch (destructive interference of wave modes), but not a reliable prediction of when it will occur, due to a high sensitivity to errors in the relative phases of the modes.

The above argument was made by Watts (1992), showing that the notch in the MCLG approximation at 4.53 kHz was caused by a 180-deg phase difference between the traveling-wave and cutoff modes, but no claim was made about a notch appearing in the numerical solution at that frequency. Parthasarathi, Grosh, and Nutall (2000) verified that indeed, no notch appears in the numerical solution at that frequency if adequate spatial sampling is used, although they showed that a notch can be artificially introduced in the numerical solution by insufficient spatial sampling. They concluded, however, that the notch in the numerical solution in a two-dimensional model is purely an artifact of insufficient spatial sampling, and thus that two-dimensional models are incapable of demonstrating the notch effect. We verified that real notches do occur in numerical solutions of the two-dimensional model [in agreement with Steele and Taber (1979) and Neely (1981)], for example in Fig. 5(a) at 1131 Hz, and thus we assert that there is no such limitation in the two-dimensional formulation.

A similar notch and change in slope was first observed by Rhode in 1971 in squirrel monkey basilar membrane responses. Rhode concluded: ''It is possible that there is another mode of vibration present in the cochlea.'' We observe that the mode-coupling Liouville–Green approximation predicts the notch and change of slope in agreement with numerical solutions, and with the same qualitative behavior as observed in biological cochleas.

However, it must be noted that the present model neglects the fast acoustic wave associated with the slight compressibility of the real cochlear fluid. Cooper and Rhode (1996) found a notch in the amplitude response above the best frequency, and concluded that it was the result of an interaction between the fast acoustic wave and the normal (nonacoustic) traveling wave mode. This finding, taken together with the present work, suggests that a notch observed in biological measurements above the best frequency could have at least two causes: an interaction of the traveling wave with the fast acoustic wave, or an interaction of the traveling wave with the cutoff mode described in this paper.

VI. CONCLUSIONS

The Liouville–Green approximation to the two-dimensional cochlear mechanics problem fails to solve Laplace's equation in the cochlear fluid. A new solution, called the mode-coupling Liouville–Green approximation, is proposed in which energy is coupled from the traveling wave mode into a second cutoff wave mode. The new approximation gives good agreement with the finite-difference solution. The second wave mode proposed in this paper may provide an explanation for the notch and change in slope of the basilar membrane responses first observed by Rhode in 1971.

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APPENDIX: THE CONVENTIONAL VALIDITY CRITERION AND ITS RELATIONSHIP TO THE RLE CRITERION

For a simple long-wave model (which applies for the cochlea near the stapes), the system is governed by the equation

$$\frac{\partial^2 \phi(x,\omega,t)}{\partial x^2} = -k^2(x)\phi(x,\omega,t), \tag{A1}$$

where

$$k(x) = \sqrt{\frac{2\rho\omega^2}{h(S(x)+i\beta(x)\omega - M\omega^2)}}. \tag{A2}$$

In the classic development (Bender and Orszag, 1978) we assume a solution of the form

$$\phi(x,\omega,t) = \phi_0 a(x) \exp i(b(x)+\omega t). \tag{A3}$$

Substituting into Eq. (A1) yields

$$k^2 - b'^2 + 2ia'b'/a + ib'' + a''/a = 0. \tag{A4}$$

By setting the first two terms equal, we get the *eikonal* equation

$$b(x) = \int_0^x k(x)\,dx, \tag{A5}$$

and by setting the second two terms equal, we get the *transport* equation

$$a(x) = k^{-1/2}(x). \tag{A6}$$

The Liouville–Green approximation amounts to neglecting the a''/a term. The Liouville–Green approximation is generally considered valid (Viergever, 1980; Zweig *et al.*, 1976) when the order of magnitude of the terms in the eikonal equation is much larger than the order of magnitude of terms in the transport equation, leading to the ''conventional validity criterion''

$$\left|\frac{k'(x)}{k(x)^2}\right| \ll 1, \tag{A7}$$

which states loosely that the wavelength should not change too fast on the scale of a single wavelength for the LG solution to be valid. This criterion becomes large near the stapes for low frequencies, and has been used to explain the poor agreement at low frequencies between the LG approximation and the numerical solution (Steele and Taber, 1982). In fact, the poor match at low frequencies is the result of using a first-order stapes displacement term, rather than the higher-order stapes displacement term of Eqs. (4) and (5) (Watts, 1992). When the higher-order term is used, the match at low frequencies is very good, as seen in Fig. 2. So, this leaves a conundrum, namely, if the LG approximation is good at low frequencies, why does the conventional validity criterion say that the LG approximation is invalid?

The reason is that the conventional validity criterion is only correct to first order. The functions $a(x)$ and $b(x)$ are designed to exactly cancel the first two pairs of terms in Eq. (A4), so the validity of the approximation cannot depend on the relative magnitudes of those terms (since they have been canceled away), but depends on the magnitude of the a''/a term relative to the largest term present, which is k^2; thus, a second-order-correct validity criterion is

$$\left|\frac{a''(x)}{a(x)k^2(x)}\right| \ll 1, \tag{A8}$$

which is the relative error introduced into Eq. (A1) by using the LG approximation. Under the scaling assumptions of Table I, in the long-wave region at low frequencies near the stapes, Eq. (A8) reduces to

$$\left|\frac{1}{16d^2k^2(x)}\right| \ll 1. \tag{A9}$$

In terms of wavelength $\lambda = 2\pi/k$, Eq. (A9) reduces to

$$\lambda \ll 8\pi d, \tag{A10}$$

which is satisfied for all the curves in Fig. 2 prior to the response peak, while Eq. (A7) is not. Note that Eq. (A9) does blow up in the limit as $k \to 0$, just much more slowly than the conventional validity criterion of Eq. (A7).

In the two-dimensional case, it can be shown that the relative error introduced into the governing equation (Laplace's equation) by the approximate solution of Eqs. (1) and (2) is given by the second-order Laplace error (SOLE)

$$\left|\frac{k'}{k^2}\left(1 - \frac{4kh(1-kh\tanh kh)}{2kh+\sinh 2kh} + 2k(y\tanh ky - h\tanh kh)\right) + \frac{a''}{ak^2}\right| \ll 1, \tag{A11}$$

that is, the relative Laplace error term (RLE) of Eq. (8) plus the second-order relative error term of Eq. (A8).

Since the reasoning behind Eqs. (A8)–(A10) leads to the conclusion that the second-order relative error term of Eq. (A8) is small, even near the stapes at low frequencies, we can drop it, leaving the two-dimensional relative Laplace error of Eq. (8) as the dominant term in the error in the governing equation

$$\left|\frac{k'}{k^2}\left(1 - \frac{4kh(1-kh\tanh kh)}{2kh+\sinh 2kh} + 2k(y\tanh ky - h\tanh kh)\right)\right| \ll 1. \tag{A12}$$

Now, finally, it can be seen that RLE$\to 0$ as $k \to 0$. This disagrees with the conventional validity criterion, which blows up quickly as $k \to 0$, but the problem lies with the conventional validity criterion, since it is not correct to second order, and does not actually measure the relative error in the governing equation. The analysis behind Eqs. (A8)–(A11) indicates that second-order error terms are small for reasonable choices of parameters such as Neely's. Thus, we conclude that the relative Laplace error accurately indicates the error in the governing equation, and that the conventional validity criterion incorrectly warns of a failure in the LG approximation near the stapes when the approximation is, in fact, a good one in that region.

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